Journal of Engineering Mathematics **33:** 251–258, 1998.

A boundary-integral equation for two-dimensional oscillatory Stokes flow past an arbitrary body

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Received 17 March 1995; accepted in revised form 10 September 1997

Abstract. The fundamental singular velocity and pressure fields generated by the presence of an isolated line force acting at a point in a two-dimensional unbounded viscous incompressible medium executing oscillatory motions are used to formulate an integral equation which governs the flow past an arbitrarily shaped body. The Fredholm integral equation of the first kind is then solved by means of a boundary-element method, for the translational oscillatory flow past circular, elliptic and orthogonally intersecting cylinders. The asymptotic behaviour of the force on the cylinder for large values of the frequency parameter is obtained.

Key words: BEM, oscillatory flow, Stokes flow, cylinders.

1. Introduction

The boundary-integral formulation for three-dimensional oscillatory Stokes flows using the singular field, has been extensively studied both from the analytical and the numerical point of view (Mc. Cracken [1], Deuring [2], and Pozrikidis [3]). It is of interest to note that, as long ago as 1896, Lorentz [4, 5, 6] derived the integral relationship between the velocity field and the continuous distribution of the fundamental singular solution (Equation 7, page 11 of [5]) which is now used so widely in the area of boundary-integral and boundary-element methods. Lorentz arrived at this equation using the well-known Stokes solution for the flow induced by a moving sphere and then taking the limits as Re $\to 0$ and $c \to \infty$ with Re $c \to 1$, where Re is the Reynolds number and c is the constant velocity of the sphere. Thus, he derives what is now known as the stokeslet, although Stokes never considered this double limit. The discovery of the stokeslet may therefore be attributed to Lorentz in 1896. Other relevant key papers in this area are Youngren and Acrivos [7], Holtsmark *et al.* [8], Loewenberg [9, 10], and Pozrikidis [11, 12, 13]. For a source of references of stokeslet solutions see [14]. Recently Liron and Barta [15] presented a new generic boundary-integral equation with physical meaning. Kim and Power [16] showed how to derive the homogeneous version of their equation from the formulation developed by Karrila and Kim [17, 18].

In this paper, we study the two-dimensional oscillatory Stokes flow in an unbounded region exterior to an arbitrarily shaped cylinder using the boundary-integral method. Streamlines and velocity profiles are presented for translational oscillatory flow past an elliptic cylinder and a cylinder with a cross section formed by two orthogonally intersecting cylinders. The asymptotic behaviour of the force exerted on the cylinder for large values of the frequency parameter is also considered.

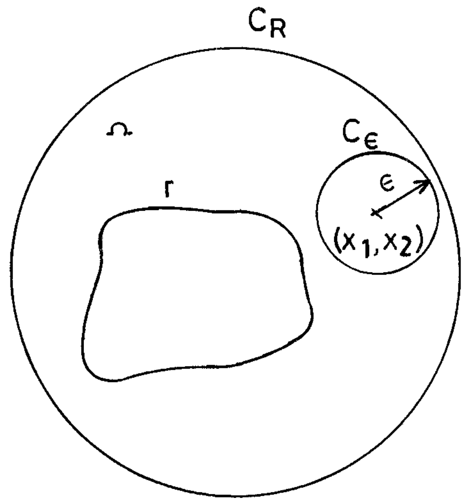


Figure 1.

2. The oscillatory line stokeslet

We consider a viscous incompressible fluid in a two-dimensional unbounded domain executing oscillatory motion. The viscous forces in the fluid are assumed to dominate the inertial forces.

The primary fundamental velocity and pressure fields (v_i, p) corresponding to an oscillatory singular line force located at the origin is given by

$$v_j = \frac{g_k}{2\pi}\left[\frac{1}{\lambda^2}\left(\frac{\delta_{kj}}{r^2} - \frac{2x_k x_j}{r^4}\right) - K_0(\lambda r)\left(\delta_{kj} - \frac{x_k x_j}{r^2}\right)\right.$$
$$\left. - \frac{K_1(\lambda r)}{\lambda r}\left(\delta_{k_j} - \frac{2x_k x_j}{r^2}\right)\right], \tag{2.1}$$

$$p = -(g_i x_i)/2\pi r^2 + P, \tag{2.2}$$

where $K_0(\lambda r)$ and $K_1(\lambda r)$ denote modified Bessel functions and P is a constant. The constant vector g_j characterises the strength and the direction of the singular line force. The Equations (2.1) and (2.2) are the primary singular velocity and pressure fields for the two-dimensional oscillatory motion. The solutions for the translational oscillations of a circular cylinder with this singular solution is given by Pop and Cheng [19].

We consider the domain Ω exterior to Γ, the contour of the cylinder, C_ε, a circle of radius ε centered at $\bar{x}$ and interior to C_R, a large circle of radius R enclosing both Γ and C_ε. The point (x_1, x_2) is an arbitrary point in this domain (see Figure 1). Using appropriate Green's identities in this domain and proceeding to the limits $\varepsilon \to 0$ and $R \to \infty$ along familiar lines, we obtain after some algebra

$$w_k(\bar{x}) = w_k^\infty(x) - \frac{1}{\mu}\int_\Gamma \phi_i(\bar{\eta}) G_{ik}(\bar{\eta}, \bar{x})\, \mathrm{d}\Gamma_\eta. \tag{2.3}$$

Here $\phi_i(\bar{\eta}) = \sigma_{ij}(\bar{\eta}) n_j(\bar{\eta})$, where σ_{ij} denotes the stress tensor and w_k^∞ denotes the undisturbed component of the flow at large distances. $G_{ik}(\bar{\eta}, \bar{x})$ denotes the field due to the stokeslet placed

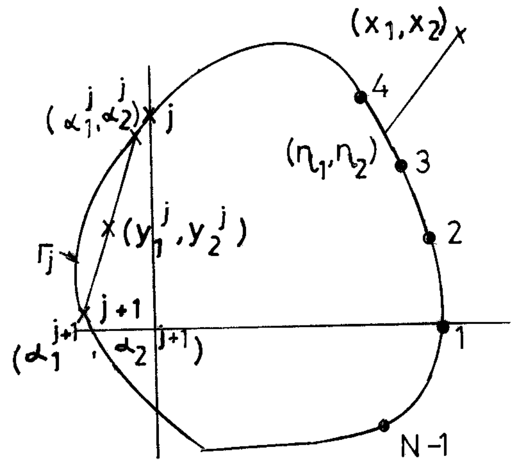


Figure 2.

at $\bar{x}$. With the help of the boundary conditions, we obtain the following system of Fredholm integral equations of the first kind for the unknown density function ϕ:

$$w_k^{\infty}(\xi_1,\xi_2) = \int_{\Gamma} \phi_i(\eta_1,\eta_2)\left\{\frac{1}{2\pi}\left[\frac{1}{\lambda_2}\left(\frac{\delta_{ki}}{\hat{r}^2}-\frac{2\hat{\xi}_k\hat{\xi}_i}{\hat{r}^4}\right)\right.\right.$$
$$\left.\left. -K_0(\lambda\hat{r})\left(\delta_{ki}-\frac{\hat{\xi}_k\hat{\xi}_i}{\hat{r}^2}\right)-\frac{K_1(\lambda\hat{r})}{\lambda\hat{r}}\left(\delta_{ki}-\frac{2\hat{\xi}_k\hat{\xi}_i}{\hat{r}^2}\right)\right]\right\}\,\mathrm{d}\Gamma_\eta, \qquad (2.4)$$

where $\hat{\xi}_i = \eta_i - \xi_i$ and $\hat{r}^2 = (\eta_i - \xi_i)(\eta_i - \xi_i)$. It is shown in the ensuing sections that the integral equation (2.4) is capable of representing any arbitrary flow with prescribed boundary conditions. It can be shown that, while the unknown density ϕ is not uniquely determined, the velocity field is uniquely determined (Power and Wrobel [20]).

3. Numerical solution of the integral equation using the boundary element technique

We divide the contour Γ into N elements and each element is denoted by Γ_j where $j = 1,2,\ldots N$, (Figure 2). Let (α_1^j,α_2^j) and $(\alpha_1^{j+1},\alpha_2^{j+1})$ denote the end points of the jth element Γ_j. Since Γ is a closed contour, we have $(\alpha_1^{n+1},\alpha_2^{n+1}) = (\alpha_1^1,\alpha_2^1)$. We now approximate each Γ_j by a linear element γ_j and the unknown function ϕ_i is assumed to be constant over each element. The constant value of ϕ_i in the jth element is assumed to be the value of ϕ_i at the midpoint of γ_j, *i.e.*, at $(\frac{1}{2}(\alpha_1^j+\alpha_1^{j+1}),\frac{1}{2}(\alpha_2^j+\alpha_2^{j+1})) = (y_1^j,y_2^j)$. Therefore the unknown quantity to be determined is $\phi_i(y_1^j,y_2^j)$ where $j = 1,2,\ldots N$. We fix the value of (ξ_1,ξ_2) to be (y_1^m,y_2^m) where m takes values $1,2,\ldots N$. With these notations (2.4) can be rewritten as

$$v_k(y_1^m,y_2^m) = \sum_{j=1}^{N}\phi_i(y_1^j,y_2^j)\int_{\gamma_j}\left\{\frac{1}{2\pi}\left[\frac{1}{\lambda_2}\left(\frac{\delta_{ki}}{\hat{r}^2}-\frac{2\hat{\xi}_k\hat{\xi}_i}{\hat{r}^4}\right)\right.\right.$$
$$\left.\left. -K_0(\lambda\hat{r})\left(\delta_{ki}-\frac{\hat{\xi}_k\hat{\xi}_i}{\hat{r}^2}\right)-\frac{K_1(\lambda\hat{r})}{\lambda\hat{r}}\left(\delta_{ki}-\frac{2\hat{\xi}_k\hat{\xi}_i}{\hat{r}^2}\right)\right]\right\}\,\mathrm{d}\Gamma_\eta \qquad (3.2)$$

where $\hat{\xi}_i = \eta_i - y_i^m$ and $r^2 = (\eta_i - y_i^m)(\eta_i - y_i^m)$. Here η_1 and η_2 are the variables of integration and $\phi_i(y_1^j, y_2^j)$ is a constant independent of (η_1, η_2).

The integral appearing in (3.2) is a regular integral for $j \neq m$. When $j = m$, the integrands possess only a removable singularity which presents no difficulty. We evaluate the integral using a 12-point Gauss quadrature formula and the resulting system of 4N × 4N linear equations are solved to obtain ϕ_i at the mid points $(y_1^j, y_2^j), j = 1, 2 \ldots N$. These values of ϕ_i are substituted in (2.3) and the integral is evaluated by means of a Gauss Quadrature formula to obtain $v_k(x_1, x_2)$, where (x_1, x_2) is any arbitrary point in the exterior of the contour Γ.

4. High frequency analysis of the force on the cylinder

In this section we consider the behaviour of the force exerted on an elliptic cylinder in the limit of high frequency oscillations. The equations of motion are formulated in terms of the stream function ψ in the elliptic coordinate system (ξ, η) defined by

$$x = c \cosh \xi \cos \eta, \qquad y = c \sinh \xi \sin \eta. \tag{4.1}$$

The physical parameters on which the force on the cylinder depends are (i) the constant value of ξ which defines the ellipse and (ii) the frequency parameter λ. The stream function ψ of the linearized flow in two dimensions satisfies the equation

$$\nabla^2(\nabla^2 - \lambda^2)\psi = 0, \tag{4.2}$$

which, in the elliptical coordinates defined by (4.1), may be split into two equations

$$\nabla_1^2 \psi = 0 \tag{4.3}$$

$$\left(\frac{1}{c^2(\cosh^2 \xi - \cos^2 \eta)} \nabla_1^2 - \lambda^2 \right) \psi = 0, \tag{4.4}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$. The solution of (4.3) which tends to zero as $\xi \to \infty$ is given by

$$\psi = \sum_{n=1}^{\infty} \exp(-n\xi)[A_n \cos(n\eta) + B_n \sin(n\eta)]. \tag{4.5}$$

The Equation (4.4) separates into the Mathieu equations for the two functions $f(\xi)$ and $f(\eta)$ where $\psi = f(\xi)g(\eta)$. To obtain the asymptotic expression for the stream function for high frequency oscillations, we transform the Mathieu equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}^2 \xi} - f(p + 2q \cosh(2\xi)) = 0, \tag{4.6}$$

first by the substitution $x = -iq^{1/2} \exp(\xi)$ and then by a transformation of the dependent variable given by $f = w(x) \exp(-x)$ to obtain

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \left(2 - \frac{1}{x}\right) \frac{\mathrm{d}w}{\mathrm{d}x} - \left(\frac{1}{x} - \frac{p}{x^2} + \frac{q^2}{x^4}\right) w = 0, \tag{4.7}$$

where $p = -\alpha_2 - \lambda^2 c^2/2$. A solution of (4.7) may be written (see [21]) as

$$w(x) = \sum_{r=0}^{\infty} (-1)^r C_r x^{-r-1/2}. \tag{4.8}$$

This gives

$$f(\xi) = \exp\{iq^{1/2}\exp(\xi)\} \sum_{r=0}^{\infty} (-1)^r C_r \{-iq^{1/2}\exp(\xi)\}^{-r-1/2}. \tag{4.9}$$

From (4.9) we obtain for, large q,

$$f(\xi) \sim \exp\{iq^{1/2}\exp(\xi)\}. \tag{4.10}$$

In a similar manner, the terms in g which dominate for large q are given by (page 41 ref. [21])

$$g(\eta) = \sin\eta + \sin(3\eta). \tag{4.11}$$

The truncated stream function corresponding to high-frequency oscillations may now be written as

$$\begin{aligned}\psi(\xi,\eta) \; = \; & K_1(\exp\{-q^{1/2}\exp(\xi)\}[\sin\eta + \sin(3\eta)]) \\ & + K_2\exp(-\xi)\sin\eta + K_3\exp(-3\xi)\sin 3\eta + Uc\sinh\xi\sin\eta. \end{aligned}\tag{4.12}$$

The constants K_1, K_2, K_3 determined by the boundary conditions to the least square approximation are

$$K_1 = -\frac{U}{2}(iq^{1/2}\exp(2\alpha) + \exp(\alpha))\exp\{-iq^{1/2}\exp(\alpha)\},$$

$$K_2 = \frac{U}{2} + \frac{U}{4}(iq^{1/2}\exp(3\alpha) + \exp(\alpha)),$$

$$K_3 = \frac{U}{20}(iq^{1/2}\exp(2\alpha) + \exp(\alpha)).$$

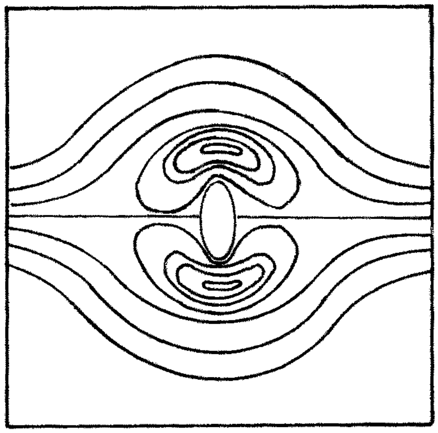
The force on the cylinder for high-frequency oscillations is given by

$$F_1 = -\frac{\lambda^2 c^2 U\pi}{2}\left(1 + \frac{\exp(\alpha)}{2}\right) + \frac{\lambda^3 c^3 U\pi}{8}\exp(2\alpha). \tag{4.13}$$

5. Results and conclusions

For a test case, we take Γ to be a circular cylinder of unit radius and divide the contour Γ into thirty linear elements. The numerical value of the velocity field obtained by solving (2.3) is in good agreement with the velocity field given by the exact solution given in [20].

We present the streamline pattern at different times at a fixed frequency $|\lambda| = 1{\cdot}0$ for an ellipse and a body composed of orthogonally intersecting cylinders. At $\omega t = 0$, the streamline pattern resembles a steady uniform flow past a cylinder. As ωt increases to $1{\cdot}35$ we see



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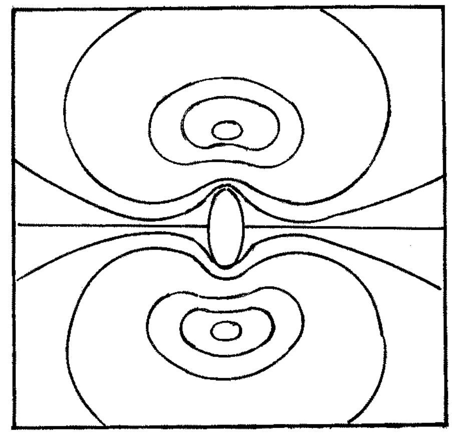
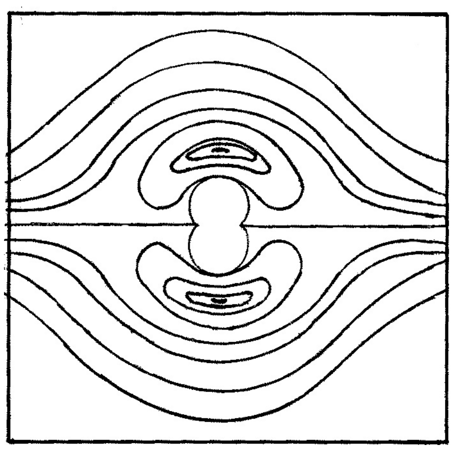


Figure 4.



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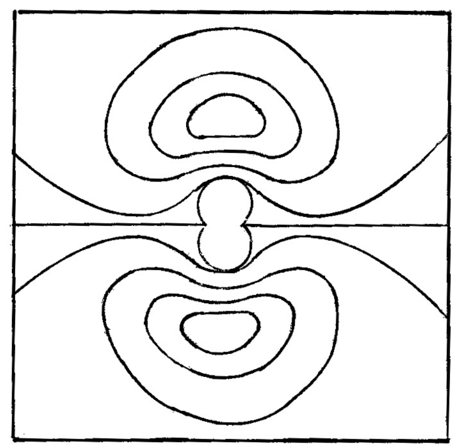

Figure 6.

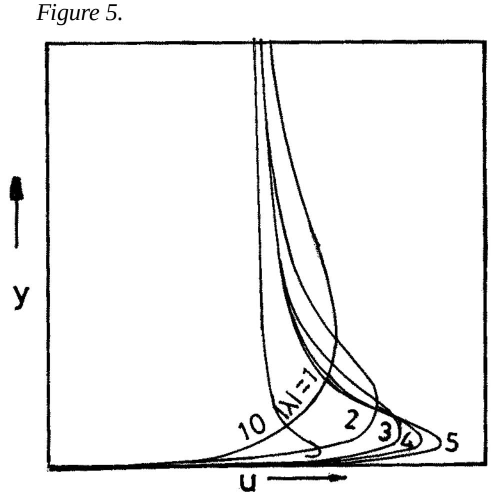

Figure 7.

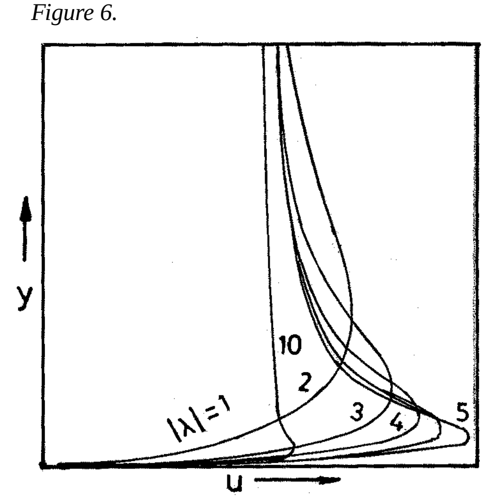

Figure 8.

eddy formations in regions very close to the cylinder (Figures 3–5) and, as time increases, the dimensions of the eddies become larger and the eddies move away from the cylinder (Figures 4–6). At still larger times the eddies vanish and the flow reaches the initial pattern. This sequence of events can be described as the formation of the eddies in the decelerating cycle and their smoothing out in the accelerating cycle.

To investigate the role played by $|\lambda|$ on the structure of the flow, we draw the velocity profiles for various values of $|\lambda|$. We present the velocity profiles at $x = 0, y \geqslant 1$ for various values of $|\lambda| > 0$. As $|\lambda|$ increases, the velocity profiles indicate the presence of a boundary layer. At $|\lambda| = 10$, we observe that in a small region near the boundary the velocity changes sharply to satisfy the no-slip conditions on the boundary (Figures 7, 8). Our results are resticted for $|\lambda| \leqslant 10$, because of computational difficulties. Because of the presence of the boundary layers for increasingly large values of $|\lambda|$, sharp changes in the velocity take place in a small region near the boundary. As mentioned at the end of section (2), the linear system of equations obtained from the Fredholm integral equation of the first kind is in general ill-conditioned. However, when the number of elements are not quite large, we obtain reasonably good solutions. It can also be inferred that the time at which the formation and smoothing of the eddies take place is the same in all cases considered.

Acknowledgement

The authors wish to thank the referees for highlighting several related references and for their constructive comments. In particular, they record their appreciation to the Chief Editor, H. K. Kuiken for bringing to the authors' attention the discovery of the stokeslet by Lorentz in 1896. One of the authors (J. Geetha) wishes to thank the Council of Scientific and Industrial Research, India for a financial grant.

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